C^*-graph algebras and beyond

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Institute for Research in Fundamental Sciences (IPM)

Quantum Groups Seminar,

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This talk is based on the following works

- Razavinia, Farrokh, and Haghighatdoost, Ghorbanali. From Quantum Automorphism of (Directed) Graphs to the Associated Multiplier Hopf Algebras. *Mathematics*, **2024**, *12.1*: 128.
- Razavinia, Farrokh. Into Multiplier Hopf ($*$-)graph algebras. *arXiv preprint arXiv: 2403.09787* (**2024**).
- Razavinia, Farrokh. A route to quantum computing through the theory of quantum graphs. **2024**, arXiv: 2404.13773.
- Razavinia, Farrokh. C^*-Colored graph algebras. *arXiv preprint* **2025** *arXiv:2504.16963.*

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And is supported with a research grant from IPM with contract No.1404140052.

- This talk will be organised as follows:

$$Multiplier\ algebras \longrightarrow multiplier\ Hopf\ algebras$$
$$\longrightarrow Graph\ algebras \longrightarrow Graph\ C^* - algebras$$
$$\longrightarrow the\ associatted\ Cuntz - Krieger\ graph\ families$$
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- At the end, I will propose some open directions.

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- and in order to proceed we try to study toy models (or better to say toy examples).
- One of these examples is the quantum $n \times n$ matrices and their coordinate ring $\mathbb{K}[M_q(n)]$,
- and the other one is an infamous set of directed graphs and the associated undirected ones, on which we partially will try to study!

Quantum matrix algebra

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- $$\begin{aligned} X_{ri}X_{rj} &= q^{-1}X_{rj}X_{ri}, && \forall i < j; \\ X_{ri}X_{si} &= q^{-1}X_{si}X_{ri}, && \forall r < s; \\ X_{ri}X_{sj} &= X_{sj}X_{ri}, && \text{if } r < s \text{ and } i > j; \\ X_{ri}X_{sj} - X_{sj}X_{ri} &= \widehat{q}X_{si}X_{rj}, && \text{if } r < s \text{ and } i < j, \end{aligned} \quad (1)$$

  where we have $\widehat{q} = q^{-1} - q$.

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- In this talk we mostly are interested in $C(G)$, the space of continuous complex valued functions on G.
- As it is already known, for G a finite group, we have $C(G) \otimes C(G) \cong C(G \times G) : f_1 \otimes f_2 \mapsto (f_1 \otimes f_2)(g_1, g_2) := f_1(g_1)f_2(g_2)$, for $f_1, f_2 \in C(G)$ and $g_1, g_2 \in G$.

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- But this no longer works when G is an infinite dimensional group, and this is exactly the place where the introduction of multiplier Hopf algebras came to assist us!

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- We know that relation $C_f(G) \otimes C_f(G) \cong C_f(G \times G)$ satisfies for this space of functions.
- In our first paper we reproved that the multiplier algebra $M(C_f(G))$ is equal to $C(G)$ for G being any group, finite or infinite.

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- Note that if already A has an identity, then $M(A) = A$.

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- Then one may define a coproduct on A as a coassociative homomorphism $\Delta : A \to M(A \otimes A)$, such that the canonical maps T_1 and T_2 defined on $A \otimes A$ by

$$T_1(a \otimes b) = \Delta(a)(1 \otimes b) \quad \text{and} \quad T_2(a \otimes b) = (a \otimes 1)\Delta(b),$$

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  have range in $A \otimes A$ and are bijective,
- and the following coassociativity condition

$$(a\otimes 1\otimes 1)(\Delta\otimes \mathrm{id})(\Delta(b)(1\otimes c)) = (\mathrm{id}\otimes\Delta)((a\otimes 1)\Delta(b))(1\otimes 1\otimes c)$$

  for all a, b and c in A and $\mathrm{id} : A \to A$ the identity map, satisfies.

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- One may see that the above definition of the coassociativity coincides with the original condition of coassociativity if the algebra has a unit.
- Then the pair (A, Δ) will be called a multiplier Hopf algebra if the maps T_1 and T_2 are bijective from $A \otimes A$ to itself.

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- Matrix $U = (u_{ij})_{i,j}$ with entries u_{ij}s from a non-trivial unital C^*-algebra satisfying relations $u_{ij} = u_{ij}^* = u_{ij}^2$ and $\sum_{k=1}^n u_{kj} = \sum_{k=1}^n u_{ik} = 1$, will be called a magic unitary.

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- This result is nice, but we been looking for something else!

Quantum permutation group S_n^+

- For $n \in \mathbb{N}$, $G = (A, u)$ will be called a compact matrix quantum group ($CMQG$) if
  1. $A = C^*(1, u_{ij}, 1 \leq i, j \leq n)$,
  2. $u = (u_{ij})_{i,j=1,\cdots,n}$, $\qquad \bar{u} = (u_{ij}^*)_{i,j=1,\cdots,n} \in M_n(A)$ are invertible,
  3. $\Delta : A \to A \otimes A : u_{ij} \mapsto \sum_{k=1}^{n} u_{ik} \otimes u_{kj}$ is a $*$-homomorphism.
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- Compact matrix quantum groups were the earliest subclass (appeared in 1987) of the class of compact quantum groups (rigorously defined in 1995).
- A compact quantum group G is a pair (A, Δ), for A a C^*-algebra and Δ a unital $*$-homomorphism from A to $A \otimes A$ satisfying in the coassociativity relation

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$$

and the cancellation properties.

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- For the compact group G, one can see $\mathbb{C}G$ as the group C^*-algebra associated with G, consisting of the set of finite linear combinations $\sum_{g \in G} c_g g$, for $c_g \in \mathbb{C}$, with the multiplication adopted from the group multiplication and equipped with the involution $(\sum c_g g)^* := \sum \overline{c}_g g^{-1}$

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- isomorphic with the universal C^*-algebra

$$C^*\left(c_g | c_g \text{ unitary},\ c_g c_h = c_{gh}, c_g^* = c_{g^{-1}}\right).$$

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- Following the above discussion, and in order to provide an answer to a question by Connes, asking

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- in the late nineties, Wang came with an answer, saying that

  “*the quantum permutation group* S_n^+

  could be defined as the largest compact

  quantum group acting on the set $\{1, \ldots, N\}$”

Quantum permutation group S_n^+, continuation

- by looking at it as the compact set $X_N := \{x_1, \ldots, x_N\}$ consisting of a finite set of points (pointwise isomorphic) and studying its function space $C(X_N) \equiv C^*\left(p_1, \cdots, p_N \text{ projections} \mid \sum_{i=1}^N p_i = 1\right)$.

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- This has led him to define $C(S_n^+)$ as the following universal C^*-algebra

$$C^*\left(u_{ij}, i, j = 1, \cdots, n \mid u_{ij} = u_{ij}^* = u_{ij}^2, \sum_{k=1}^n u_{kj} = \sum_{k=1}^n u_{ik} = 1\right)$$

Quantum permutation group S_n^+, continuation

- and calling $S_n^+ = (C(S_n^+), u)$ the quantum symmetric (permutation) group as the quantum automorphism group of X_N, and proving that it satisfies the relations of being a compact (matrix) quantum group in the sense of Woronowicz.

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- The main ingredients in defining $C(S_n^+)$, meaning that the u_{ij}s, are very important in our construction of the multiplier Hopf ($*$-)graph algebras.

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- But what are C^*-graph algebras?
- To deal with them, one first needs to know about the Cuntz-Krieger graph families.

Cuntz-Krieger Algebras

- To a directed graph Γ, one can associate a C^*-algebra $C^*(\Gamma^0, \Gamma^1) := C^*(\Gamma)$ by associating to its set of edges Γ^1 a set of partial isometries and to its set of vertices Γ^0 a set of pairwise orthogonal projections satisfying in some specific relations, studied first by Cuntz and Krieger in 1980, as a generalization of the Cuntz algebras.

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- For a finite or infinite-dimensional Hilbert space $\mathcal{H}$, the set of mutually orthogonal projections $p_v \in \mathcal{H}$ for all $v \in \Gamma^0$ together with partial isometries $s_e \in \mathcal{H}$ for all $e \in \Gamma^1$ satisfying the relations

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- For $n \times n$ matrix $\Pi \in M_n(\{0,1\})$, the Cuntz-Krieger algebra $\mathcal{K}_\Pi$ will be defined as the (nondegenerate) C^*-algebra generated by a universal Cuntz-Krieger Γ-family s_i for $i \in \{1, \cdots, n\}$ satisfying in $s_i^* s_i = \sum_{j=1}^{n} a_{ij} s_j s_j^*$.

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- For finite directed graph $\Gamma = (\Gamma^0, \Gamma^1)$, the graph C^*-algebra $\mathcal{C}^*(\Gamma)$ is the universal C^*-algebra generated by a Cuntz-Krieger Γ-family $\{P_v, S_e\}$.

Quantum matrix algebra

- It is already known that one can associate a directed graph to the set of defining relations of $\mathbb{K}[M_q(n)]$ by using the following rules:
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- and we have $u_{ij} \overset{\leftrightarrow}{\sim} u_{k\ell}$ if and only if $i > k$ and $j < \ell$.

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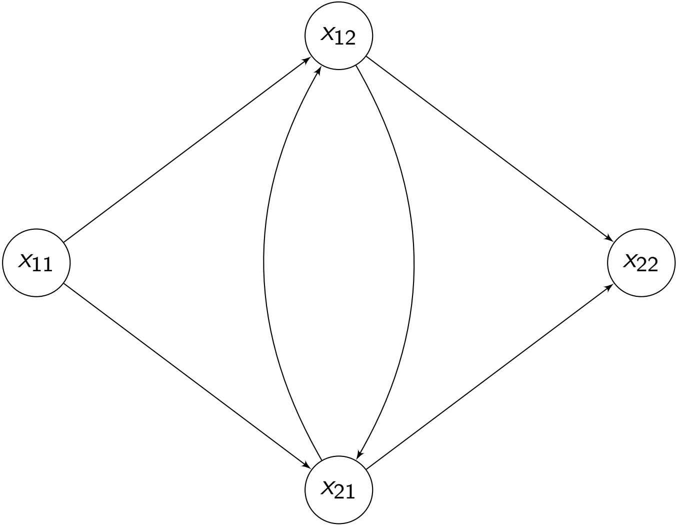
- 

Figure 1: **Directed locally connected graph related to $\mathbb{K}\,(M_q(2))$**

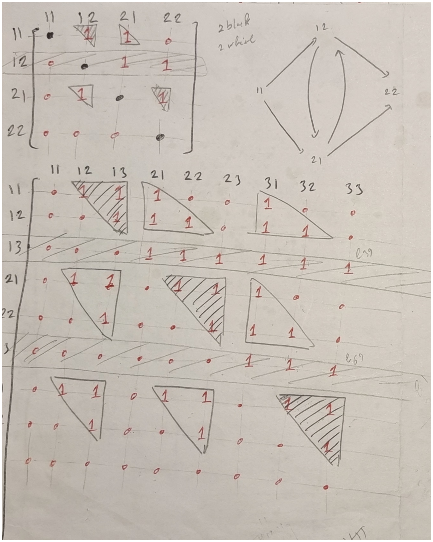


Figure 2: Illustration of the some sort of triangulation in $\mathcal{G}_2$ and $\mathcal{G}_3$

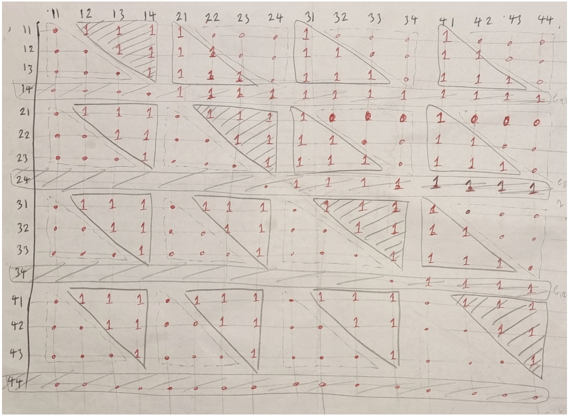


Figure 3: Illustration of the some sort of triangulation in $\mathcal{G}_4$

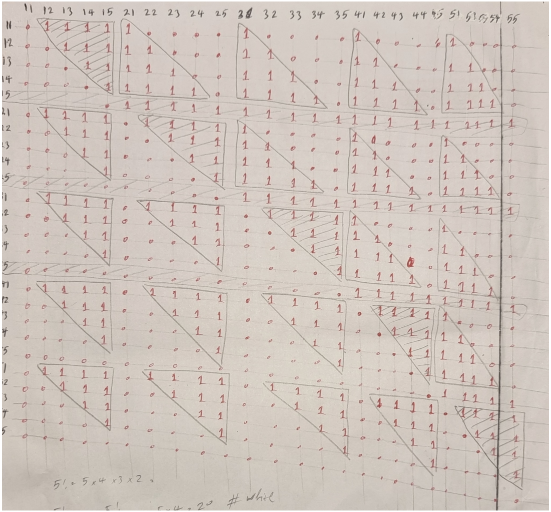


Figure 4: Illustration of the some sort of triangulation in $\mathcal{G}_5$

Graph C^*-algebra of a finite directed graph

- For graph $\mathcal{G}_2$ associated with $\mathbb{K}[M_q(2)]$, consider its set of vertices and edges as $\mathcal{G}^0 = \{x_{11} := u, x_{12} := v, x_{22} := k, x_{21} := w\}$ and $\mathcal{G}^1 = \{x_{11}\overrightarrow{\sim}x_{12} := e, x_{11}\overrightarrow{\sim}x_{21} := f, x_{12}\overrightarrow{\sim}x_{22} := h, x_{21}\overrightarrow{\sim}x_{22} := g, x_{12}\overrightarrow{\sim}x_{21} := i, x_{21}\overrightarrow{\sim}x_{12} := j\}$, we have the following Proposition.

Graph C^*-algebra of a finite directed graph

- For graph $\mathcal{G}_2$ associated with $\mathbb{K}[M_q(2)]$, consider its set of vertices and edges as $\mathcal{G}^0 = \{x_{11} := u, x_{12} := v, x_{22} := k, x_{21} := w\}$ and $\mathcal{G}^1 = \{x_{11} \overset{\rightarrow}{\rightsquigarrow} x_{12} := e, x_{11} \overset{\rightarrow}{\rightsquigarrow} x_{21} := f, x_{12} \overset{\rightarrow}{\rightsquigarrow} x_{22} := h, x_{21} \overset{\rightarrow}{\rightsquigarrow} x_{22} := g, x_{12} \overset{\rightarrow}{\rightsquigarrow} x_{21} := i, x_{21} \overset{\rightarrow}{\rightsquigarrow} x_{12} := j\}$, we have the following Proposition.
- For Π_2 as before, and $\mathcal{G}_2 = (\mathcal{G}_2^0, \mathcal{G}_2^1)$ the associated adjacency matrix, and let $\mathcal{H} := \ell^2(\mathbb{N})$ be the underlying infinite dimensional Hilbert space. Then the set

$$\begin{aligned} S = \{S_e &:= \sum_{n=1}^{\infty} E_{6n,3n-2}, S_f := \sum_{n=1}^{\infty} E_{6n-4,3n-2}, S_h := \sum_{n=1}^{\infty} E_{6n-3,3n}, \\ S_g &:= \sum_{n=1}^{\infty} E_{6n-4,3n-1}, S_i := \sum_{n=1}^{\infty} E_{6n-1,3n}, S_j := \sum_{n=1}^{\infty} E_{6n-3,3n-1}\} \end{aligned} \tag{2}$$

is a Cuntz-Krieger $\mathcal{G}(\Pi_2)$-family and gives us an infinite dimensional graph C^*-algebra structure $\mathcal{C}^*(\Pi_2)$.

Graph C^*-algebra of a finite directed graph

- Let π_n be the commuting matrices with Π_n and ${}_n\mathcal{G}$ be the associated graphs.

Graph C^*-algebra of a finite directed graph

- Let π_n be the commuting matrices with Π_n and ${}_n\mathcal{G}$ be the associated graphs.
- For graph ${}_2\mathcal{G}$ associated with π_2, as before consider its set of vertices and edges as ${}_2\mathcal{G}^0 = \{x_{11} := v_1, x_{12} := v_2, x_{22} := v_3, x_{21} := v_4\}$ and ${}_2\mathcal{G}^1 = \{x_{11} \overrightarrow{\sim} x_{11} := e_{11}, x_{12} \overrightarrow{\sim} x_{21} := e_{24}, x_{21} \overrightarrow{\sim} x_{12} := e_{42}, x_{22} \overrightarrow{\sim} x_{22} := e_{33}\}$, we have the following Proposition.

Graph C^*-algebra of a finite directed graph

- Let π_n be the commuting matrices with Π_n and ${}_n\mathcal{G}$ be the associated graphs.
- For graph ${}_2\mathcal{G}$ associated with π_2, as before consider its set of vertices and edges as ${}_2\mathcal{G}^0 = \{x_{11} := v_1, x_{12} := v_2, x_{22} := v_3, x_{21} := v_4\}$ and ${}_2\mathcal{G}^1 = \{x_{11} \overrightarrow{\sim} x_{11} := e_{11}, x_{12} \overrightarrow{\sim} x_{21} := e_{24}, x_{21} \overrightarrow{\sim} x_{12} := e_{42}, x_{22} \overrightarrow{\sim} x_{22} := e_{33}\}$, we have the following Proposition.
- For ${}_2\mathcal{G} := ({}_2\mathcal{G}^0, {}_2\mathcal{G}^1)$ as above, consider $\mathcal{H}$ be the underlying Hilbert space, that can be finite or infinite. Then the set

$$S = \{S_{e_{11}} := E_{2,1}, S_{e_{24}} := E_{4,1}, \\ S_{e_{42}} := E_{1,4}, S_{e_{33}} := E_{3,1}\} \tag{3}$$

is a Cuntz-Krieger ${}_2\mathcal{G}$-family and gives us a graph C^*-algebra structure $\mathcal{C}^*(\pi_2) := M_4(\mathbb{C})$.

Multiplier Hopf $*$-graph algebras

- As I already said, from the different kinds of graph algebras, the one that we are interested in should be nondegenerate, and we thought that the $*$-monoid algebra $\mathcal{G}$ consisting of graphs associated with π_n the commuting commutative matrices with Π_n's, and the identity element ${}_2\mathcal{G}$, illustrated in Figure 5.

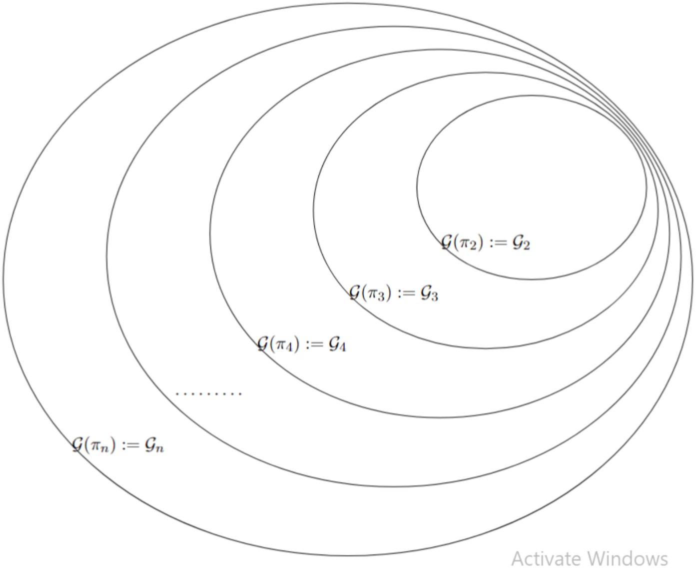


Figure 5: Illustration of the set of $n-1$ graphs ${}_i\mathcal{G}$

Multiplier Hopf $*$-graph algebras

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Multiplier Hopf $*$-graph algebras

- But the question was how?
- At this point we had a slightly smooth result based on the work of Rollier-Vaes, as follows

Multiplier Hopf $*$-graph algebras

- For Π, a locally finite connected graph associated with coordinate algebra $\mathbb{K}(M_q(n))$ with vertex set $\{x_{11}, x_{12}, \cdots, x_{ij}\}$ for $i, j \in \{1, 2, \cdots, n\}$ and the index set $I := \{11, 12, \cdots, ij\}$, there exists a unique universal nondegenerate $*$-algebra $\mathcal{A}$ generated by elements $(u_{hh'})_{h,h' \in I}$, satisfying the relations of quantum permutation groups, and a unique nondegenerate $*$-homomorphism $\Delta : \mathcal{A} \to M(\mathcal{A} \otimes \mathcal{A})$ satisfying $\Delta(u_{hh'}) = \sum_{k \in I}(u_{hk} \otimes u_{kh'})$ for all $h, h' \in I$, such that the pair $(\mathcal{A}, \Delta)$ is a multiplier Hopf $*$-algebra in the sense of Van Daele,
- But we have not been satisfied with this result!

Multiplier Hopf $*$-graph algebras

- Let us invite back the vector space $\mathcal{G}$, consisting of the $(n-1)$- locally finite graphs ${}_i\mathcal{G}$, to the scene.

Multiplier Hopf $*$-graph algebras

- Let us invite back the vector space $\mathcal{G}$, consisting of the $(n-1)$- locally finite graphs ${}_i\mathcal{G}$, to the scene.
- This is a unital $*$-monoid algebra, and in order to have a $*$-multiplier Hopf algebra, we need to define a map Δ on $\mathcal{G}$ to $M(\mathcal{G} \otimes \mathcal{G})$, resembling the co-product and satisfying the co-associativity condition

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- $$(\pi_i \otimes 1 \otimes 1)(\Delta\otimes)(\Delta(\pi_j)(1\otimes\pi_k)) \\ = \\ (\otimes\Delta)((\pi_i\otimes 1)\Delta(\pi_j))(1\otimes 1\otimes\pi_k),$$

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- Such that $\Delta(\pi_i)(1 \otimes \pi_j)$ and $(\pi_i \otimes 1)\Delta(\pi_j)$ belong in $\mathcal{G} \otimes \mathcal{G}$, for any $\pi_i, \pi_j, \pi_k \in \mathcal{G}$.

Multiplier Hopf $*$-graph algebras

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Multiplier Hopf $*$-graph algebras

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- How can we define such a map for a graph algebra consisting of graphs?

Multiplier Hopf $*$-graph algebras

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- As the previous constructions suggest, the idea is to implement the graph C^*-algebra $C^*(S, P)$ associated with π_is. In this case, we will obtain n different multiplier Hopf $*$-graph algebras.

Multiplier Hopf $*$-graph algebras

- But now the question was that
- How can we define such a map for a graph algebra consisting of graphs?
- As the previous constructions suggest, the idea is to implement the graph C^*-algebra $C^*(S, P)$ associated with π_is. In this case, we will obtain n different multiplier Hopf $*$-graph algebras.
- And we have the following result concerning the first initial examples of the multiplier Hopf $*$-graph algebras.

Multiplier Hopf $*$-graph algebras

- For the graph C^*-algebra $C^*(S,P) := C^*(\pi_n) = M_{n^2}(\mathbb{C})$, and the Cuntz-Krieger ${}_n\mathcal{G}$-family

$$S = \begin{cases} S_{e_{ii}} := E_{i+1,1} & \\ S_{e_{ij}} := E_{j,1} & \text{for } j \geq i \\ S_{e_{ij}} := E_{1,i} & \text{for } i \geq j \end{cases},$$

Multiplier Hopf $*$-graph algebras

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- define

$$\Delta : \mathcal{O}(M_{n^2}(\mathbb{C}))[t^{-1}] \to M(\mathcal{O}(M_{n^2}(\mathbb{C}))[t^{-1}] \otimes \mathcal{O}(M_{n^2}(\mathbb{C}))[t^{-1}]) \tag{4}$$

$$E_{i,j} \longmapsto E_{k,h} \otimes E_{o,r} := E_{\ell,m}, \tag{5}$$

for $\ell = P_o^k$ and $m = P_r^h$, expanded linearly on whole of $\mathcal{O}(M_{n^2}(\mathbb{C}))[t^{-1}]$. Then Δ is a coproduct on $\mathcal{O}(M_{n^2}(\mathbb{C}))[t^{-1}] = \mathcal{O}(Gl(n))$, and $(\mathcal{O}(Gl(n)), \Delta)$ is a multiplier Hopf $*$-graph algebra, for $i, j, k, h, o, r \in \{1, \cdots, n^2\}$ and $\ell, m \in \{1, \cdots, 2n^2\}$.

Multiplier Hopf $*$-graph algebras

- Also there exists a unique linear map $\epsilon : \mathcal{O}(GL(n)) \to \mathbb{C}$ taking $E_{i,j}$ to δ_{ij} such that

$$(\epsilon\otimes)(\Delta(E_{k,\ell})(1 \otimes E_{o,r})) = E_{k,\ell}E_{o,r} \tag{6}$$

$$(\otimes\epsilon)((E_{k,\ell} \otimes \Delta(E_{o,r})) = E_{k,\ell}E_{o,r}, \tag{7}$$

for all $E_{o,r}, E_{k,\ell}$ associated to all $X_{o,r}, X_{k,\ell} \in \mathcal{O}(GL(n))$, and ϵ is a homomorphism.

Multiplier Hopf ∗-graph algebras

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  for all $E_{o,r}, E_{k,\ell}$ associated to all $X_{o,r}, X_{k,\ell} \in \mathcal{O}(GL(n))$, and ϵ is a homomorphism.
- There is a unique linear map $S : \mathcal{O}(GL(n)) \to M(\mathcal{O}(GL(n)))$ taking $E_{i,j}$ to $E_{j,i}$, associated with $X_{i,j}$ and $X_{j,i}$ respectively, such that

$$m(S\otimes)(\Delta(E_{o,r})(1 \otimes E_{k,\ell})) = \epsilon(E_{o,r})E_{k,\ell} \tag{8}$$
$$m(\otimes S)((E_{o,r} \otimes 1)\Delta(E_{k,\ell})) = \epsilon(E_{k,\ell})E_{o,r}, \tag{9}$$

  for all $E_{o,r}, E_{k,\ell}$ as above, and m denotes multiplication, defined as a linear map from $M(\mathcal{O}(GL(n))) \otimes \mathcal{O}(GL(n))$ to $\mathcal{O}(GL(n))$ and from $\mathcal{O}(GL(n)) \otimes M(\mathcal{O}(GL(n)))$ to $\mathcal{O}(GL(n))$. The map S is an anti-homomorphism.

The second toy example

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The second toy example

- Let us have some colors in our mathematics!
- In our study, the vertices of our graphs will be colored in three colors red, blue, and green.
- It means that the vertex chromatic number will be 3.

The second toy example; Continuation

- Consider the following directed colored simple graph, which is almost the same as Π_2, but undirected and colored

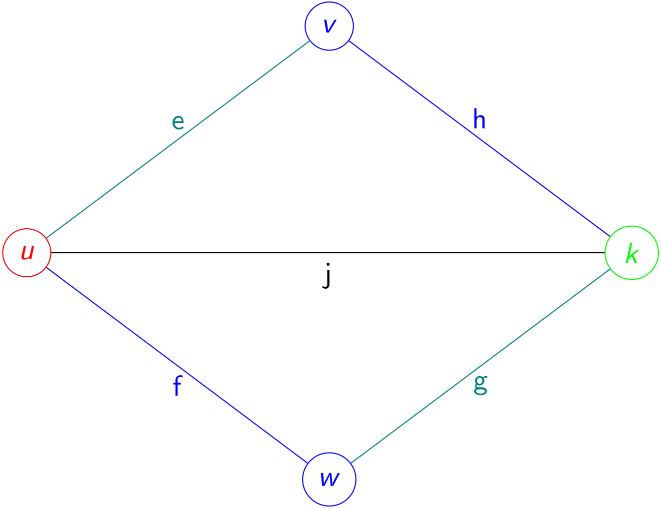

Figure 6: **Two connected graph** Sq_2

The second toy example; Continuation

- Let us try to enlarge Sq_2 in a very structural way and to obtain Sq_3.

The second toy example; Continuation

- Let us try to enlarge Sq_2 in a very structural way and to obtain Sq_3.
- Note that, for Sq_3 we have, $\chi_v(\mathrm{Sq}_3) = 3$, $K(\mathrm{Sq}_3) = 4$, and the edge chromatic number $\chi_e(\mathrm{Sq}_3) = 4$.

The second toy example; Continuation

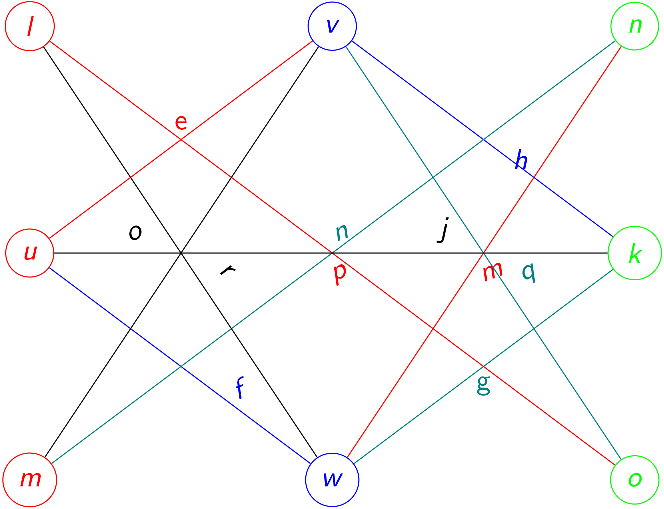

Figure 7: **Four connected graph** Sq_3

The second toy example; Continuation

- Let us move one more step further and create Sq_4.

The second toy example; Continuation

- Let us move one more step further and create Sq_4.
- Note that, for Sq_4 we have, $\chi_v(\mathsf{Sq}_4) = 3$, $K(\mathsf{Sq}_4) = 6$, and the edge chromatic number $\chi_e(\mathsf{Sq}_4) = 6$.

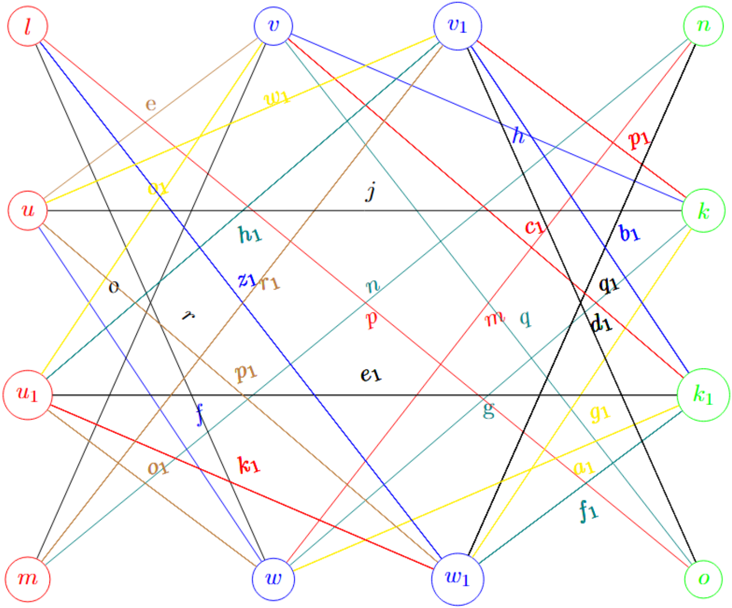

Figure 8: **Six connected graph** Sq_4

The second toy example; Continuation

- Note that the process of enlarging the set of graphs Sq_i will continue, and we call this special set of graphs with $\mathsf{G}_s = \{\mathsf{Sq}_i | i \geq 2\}$.

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- Γ_0 will be the null graph, the graph with no edges.
- let $\Gamma = (\Gamma^0, \Gamma^1) \in \mathrm{G}_s / \{\Gamma_0\}$ be a graph with set of vertices Γ^0 and set of edges Γ^1, such that $\#\Gamma^0 = n^2 - (n-2)^2$.

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- Then for $n \in \{3, 4, \cdots\}$, the number of edges will be

$$\#\Gamma^1 = 1 + (n-1)(4n-3), \tag{10}$$

and the number of Hamiltonian paths will be

$$\#\mathscr{H}_\Gamma = 10n + (2n-1)(2n-9). \tag{11}$$

Graph colored algebra

- For $i \in \{3, 4, \cdots\}$, the graphs in $\mathsf{G}_{s_i} := \mathsf{Sq}_i$ will consists of two layers. The inner, which is a $(i-2) \times (i-2)$ lattice array of vertices, and the outer layer, which is a 2×2 lattice array.

Graph colored algebra

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- Let V_b, V_g, V_r be the set of blue, green, and red vertices, respectively, and let Γ_1, Γ_2 be in G_s.

Graph colored algebra

- Then the connect and overlay operators will be defined as follows for graphs $\Gamma_1 = (\Gamma_1^0, \Gamma_1^1)$ and $\Gamma_2 = (\Gamma_2^0, \Gamma_2^1)$.

$$\Gamma_1 + \Gamma_2 := \left(\Gamma_1^0 \cup \Gamma_2^0, \Gamma_1^1 \cup \Gamma_2^1\right) \tag{12}$$

$$\Gamma_1 \to \Gamma_2 := \left(\Gamma_1^0 \cup \Gamma_2^0, (\Gamma_1^1 \cup \Gamma_2^1)/\{V_b \to V_b \ \&\ V_r \to V_r \ \&\ V_g \to V_g\}\right. \tag{13}$$

$$\Gamma_0 \to \Gamma := (\Gamma^0, \Gamma^1), \qquad \forall \Gamma = (\Gamma^0, \Gamma^1) \in \mathsf{G}_s \text{ and } \Gamma_0 \text{ the null graph.} \tag{14}$$

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- Then, the set G_s will have a unital nondegenerate $*$-monoid algebra structure equipped with the above binary operations, together with the identity element Γ_0, and the diagrammatic illustration as in Figure 5, the illustration of the set of $n-1$ graphs ${}_i\mathcal{G}$.

C^*-colored graph algebras

- In order to move to the C^*-graph algebra case, we need to work with the directed graphs.

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- Consider the outer layer with $\mathscr{L}_o$ and the inner layer with $\mathscr{L}_i$.
- Consider the red and green vertices in the outer (and inner layer) with $r_{\mathscr{L}_o}$ ($r_{\mathscr{L}_i}$) and $g_{\mathscr{L}_o}$ ($r_{\mathscr{L}_i}$) respectively.

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- In order to move to the C^*-graph algebra case, we need to work with the directed graphs.
- Consider the outer layer with $\mathscr{L}_o$ and the inner layer with $\mathscr{L}_i$.
- Consider the red and green vertices in the outer (and inner layer) with $r_{\mathscr{L}_o}$ ($r_{\mathscr{L}_i}$) and $g_{\mathscr{L}_o}$ ($r_{\mathscr{L}_i}$) respectively.
- Consider the set of red and green vertices with $O_r := \{O_{r_j} \mid j \in \{1, \cdots, i-2\}\}, O_g := \{O_{g_j} \mid j \in \{1, \cdots, i-2\}\}, I_r := \{I_{r_j} \mid j \in \{1, \cdots, i-2\}\}$, and $I_g := \{I_{g_j} \mid j \in \{1, \cdots, i-2\}\}$, in outer and inner layers, respectively.

C^*-colored graph algebras; Continuation

- Then, the orientation of the graphs in G_s will be naturally defined as follows.

C^*-colored graph algebras; Continuation

- Then, the orientation of the graphs in G_s will be naturally defined as follows.
- 1. The vertices of the graph $\Gamma \in \mathrm{G}_s$ will be connected unless they are in the same color category.
  2. $r_{\mathscr{L}_i} \leftrightarrow g_{\mathscr{L}_i}$ & $r_{\mathscr{L}_o} \leftrightarrow g_{\mathscr{L}_o}$,
  3. $I_{g_j} \rightarrow O_{r_{j'}}$ & $O_{r_{j''}} \rightarrow I_{g_j}$,
  4. $I_{r_j} \rightarrow O_{g_{j'}}$ & $O_{g_{j''}} \rightarrow I_{r_j}$.

C^*-colored graph algebras; Continuation

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  4. $I_{r_j} \rightarrow O_{g_{j'}}$ & $O_{g_{j''}} \rightarrow I_{r_j}$.
- for $j'' \neq j \neq j'$ (meaning that the above connections are between different vertices with different colors).

C^*-colored graph algebras; Continuation

- For example, Sq_3 will change as follows and will be called Sq_3^d.

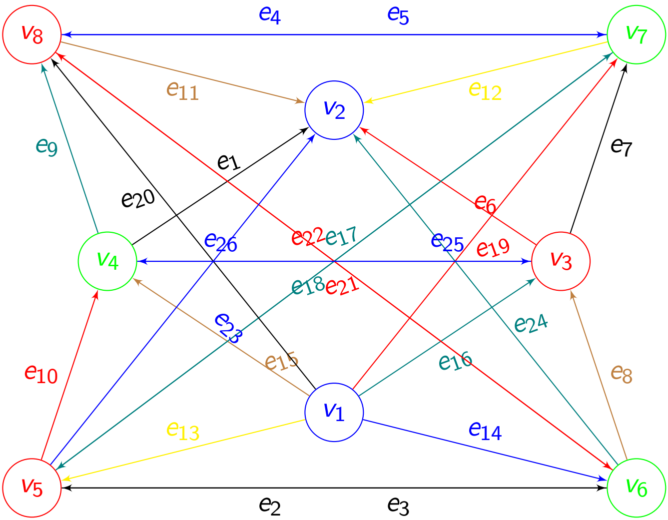

Figure 9: **Five connected directed graph** Sq_3^d

C^*-colored graph algebras; Continuation

- Sq_3^d has 28 Hamiltonian paths.

C^*-colored graph algebras; Continuation

- Sq_3^d has 28 Hamiltonian paths.
- The plan is to associate each special edge involved in each special Hamiltonian path a partial (matrix) isometry $\mathscr{S}_i$ and to each vertex an orthogonal projection $\mathscr{P}_i$,

C^*-colored graph algebras; Continuation

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- and look for possible graph Cuntz-Krieger families, and to see if they produce infinite or finite graph C^*-algebras.

C^*-colored graph algebras; Continuation

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- and look for possible graph Cuntz-Krieger families, and to see if they produce infinite or finite graph C^*-algebras.
- For some reason we mostly prefer the finite ones!

C^*-colored graph algebras; Continuation

- But before doing so, let us have a look at Sq_4^d.

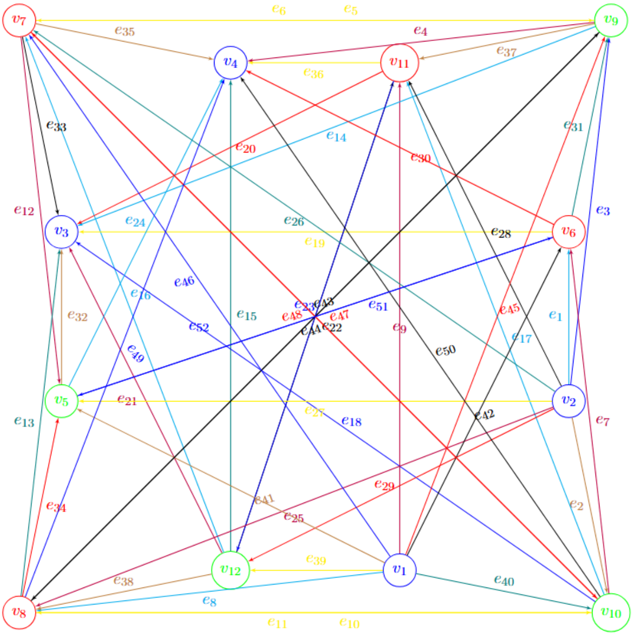

Figure 10: **Seven connected directed graph** Sq_4^d

C^*-colored graph algebras; Continuation

- Sq_4^d has 216 Hamiltonian paths.

C^*-colored graph algebras; Continuation

- Sq_4^d has 216 Hamiltonian paths.
- For $n \in \{3, 4, \cdots\}$, there are $n - 2$ sink and source vertices in Sq_n^d.

C^*-colored graph algebras; Continuation

- Sq_4^d has 216 Hamiltonian paths.
- For $n \in \{3, 4, \cdots\}$, there are $n-2$ sink and source vertices in Sq_n^d.
- and number of edges and vertices in Sq_n^d is equal to $26(n-2)$, and $4(n-1)$, respectively.

C^*-colored graph algebras; Continuation

- Sq_4^d has 216 Hamiltonian paths.
- For $n \in \{3, 4, \cdots\}$, there are $n-2$ sink and source vertices in Sq_n^d.
- and number of edges and vertices in Sq_n^d is equal to $26(n-2)$, and $4(n-1)$, respectively.
- and Sq_n^d has the vertex chromatic number $\chi_v(\mathrm{Sq}_n^d) = 3$, and the edge chromatic number $\chi_e(\mathrm{Sq}_n^d) = 2n$.

C^*-colored graph algebras; Continuation

- One might be interested in proving that the number of Hamiltonian paths of Sq_n^d for $n \in \{3, 4, \cdots\}$ is as follows

$$\#\mathscr{H}_{\mathrm{Sq}_n^d} = 7(n+1) + 188(n-3). \tag{15}$$

C^*-colored graph algebras; Continuation

- As a reminder, let us recall the following statement, stated as a claim in one of our papers and then proved in our next paper.

C^*-colored graph algebras; Continuation

- As a reminder, let us recall the following statement, stated as a claim in one of our papers and then proved in our next paper.
- For $\mathcal{G}_n = (\mathcal{G}_n^0, \mathcal{G}_n^1)$ the associated directed locally finite graphs with $\mathbb{K}[M_q(n)]$, and Π_n the associated adjacency matrices, and $\mathcal{H} := \ell^2(\mathbb{N})$ the underlying infinite dimensional Hilbert space. The claim is that the set S, defined as

$$\{S_i \mid \text{for fixed } 1 \leq i \leq \frac{(n^3+n^2)(n-1)}{2}\},$$

for $S_i := \sum_{j=1}^{\infty} {}^iE_{\mathcal{E}j-A,(n^2-1)j-D}$, is a Cuntz-Krieger $\mathcal{G}_n$-family for $D \in \{0, \cdots, n^2-2\}$, and $\mathcal{E}$ depends on the degree of the exit edges to the vertex e_{hk}, where i is considered as an exit edge, i.e. if $\overrightarrow{\deg}_{hk} = 2$, then we will have $\mathcal{E} = 2(n^2-1)$, and if it is 3, then we will have $\mathcal{E} = 3(n^2-1)$, and so on, and $A \in \{0, \cdots, \overrightarrow{\deg}_{hk} \times (n^2-1)\}$, and gives us a graph C^*-algebra structure $\mathcal{C}^*(\mathcal{G}_n)$.

C^*-colored graph algebras; Continuation

- Then we have the following claim and a minor result after that.

C^*-colored graph algebras; Continuation

- Let $n \in \{3, 4, \cdots\}$, and $\Gamma = (\Gamma^0, \Gamma^1)$ be an arbitrary colored directed graph with chromatic vertex number $\chi_v(\Gamma)$ and $|\Gamma^0| = (\chi_v + 1)(n-1)$, with $n - \chi_v + 1$ equal number of sink and source vertices. And let $\mathcal{A}_n$ be the associated adjacency matrix, and $\mathcal{H} := \ell^2(\mathbb{N})$ be the underlying infinite dimensional Hilbert space. Then the claim is that the set

$$S = \{S_i \mid \text{ for fixed } 1 \leq i \leq 26(n - \chi_v + 1)\},$$

for $S_i := \sum_{j=1}^{\infty} {}^{i}E_{\mathcal{E}j-A,(n^2-1)j-D}$, is a Cuntz-Krieger $\mathcal{G}$-family for $D \in \{0, \cdots, n^2-2\}$, and $\mathcal{E}$ depends on the degree of the exit edges of the vertex e_{hk}, where i is considered as an exit edge, i.e. if $\overrightarrow{\deg}_{hk} = 2$, then we will have $\mathcal{E} = 2(n^2-1)$, and if it is 3, then we will have $\mathcal{E} = 3(n^2-1)$, and so on, and $A \in \{0, \cdots, \overrightarrow{\deg}_{hk} \times (n^2-1)\}$.
S gives us an infinite-dimensional graph C^*-algebra structure $\mathcal{C}^*(\Gamma)$.

C^*-colored graph algebras; Continuation

- And we have the following immediate result.

C^*-colored graph algebras; Continuation

- And we have the following immediate result.
- For Sq_3^d, and its adjacency matrix $\mathcal{A}_3$, let $\mathcal{H} := \ell^2(\mathbb{N})$ be the underlying infinite dimensional Hilbert space. Then the set

$$S = \{S_{e_i} := \sum_{j=1}^{\infty} {}^{i}E_{\mathcal{E}j-A,8j-D} \mid \text{ for fixed } 1 \leq i \leq 26\},$$

  is a Cuntz-Krieger $\mathcal{A}_3$-family for $D \in \{0, \cdots, 7\}$, $\mathcal{E} \in \{16, 24, 32, 48\}$, and $A \in \{0, \cdots, \mathcal{E}\}$ depending on the chosen edges, and gives us the graph C^*-algebra structure $\mathcal{C}^*(\mathrm{Sq}_3^d)$.

Looking for a solution

- Not all graphs have quantum symmetry, as not all of them are symmetrical.

Looking for a solution

- Not all graphs have quantum symmetry, as not all of them are symmetrical.
- We say a graph Γ possesses quantum symmetries, or in other words, its quantum automorphism group $G_{\mathsf{QAut}}(\Gamma)$ is not trivial if there exists a unique noncommutative magic unitary matrix $u = (u_{ij})_{i,j}$ such that we have $u\mathcal{A}_\Gamma = \mathcal{A}_\Gamma u$.

Looking for a solution

- One may verify that, the following matrix

$$
{}^{c}\mathcal{A}_3^d = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & p & 0 & 0 & 1-p & 0 \\
0 & 0 & 0 & 0 & q & 1-q & 0 & 0 \\
0 & 1-p & 0 & 0 & 0 & 0 & p & 0 \\
0 & 0 & 1-q & 0 & 0 & q & 0 & 0 \\
0 & 0 & q & 0 & 1-q & 0 & 0 & 0 \\
0 & p & 0 & 1-p & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
$$

  commutes with $\mathcal{A}_3^d$, the adjacency matrix of Sq_3^d only if we have $p = q$.

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- So, $\mathcal{A}_3^d$ does not possess any quantum symmetries, and for $p \in \{0, 1\}$ we will get two different commuting matrices!

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  commutes with $\mathcal{A}_3^d$, the adjacency matrix of Sq_3^d only if we have $p = q$.
- So, $\mathcal{A}_3^d$ does not possess any quantum symmetries, and for $p \in \{0, 1\}$ we will get two different commuting matrices!
- And this is true for all graph s Sq_n^d for $n \in \{3, 4, \cdots\}$!

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- Concerning what I said, there are many open directions to tackle.
- The first question is about proceeding with multiplier Hopf ($*$-)graph algebras by just employing the graph algebra structures consisting of (directed) graphs. Which is not an easy job to consider!
- Another direction could be proposed by using the generalized definition of the magic unitary matrices. It might be included in our next work!
- Another interesting direction or better to say question, is to find a graph, directed or undirected, with trivial automorphism group and nontrivial quantum automorphism group!

Concluding Remarks

- Looking for another sitting quantum group between S_n and S_n^+ could be an interesting study to consider. For example, for $n = 4$ and $n = 5$, it is known that there are no such intermediate quantum groups, but for the other cases the answer is still unknown!

Concluding Remarks

- Looking for another sitting quantum group between S_n and S_n^+ could be an interesting study to consider. For example, for $n = 4$ and $n = 5$, it is known that there are no such intermediate quantum groups, but for the other cases the answer is still unknown!
- But one of my fantasies is looking for a graph or a set of graphs with their automorphism groups equal to A_n the alternating group. And then try to find if they possess quantum symmetries or not. If they have, then try to see what is it, and just call it A_n^+, or just prove that such kind of graphs don't have quantum symmetries, and so there is nothing to call it A_n^+!

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Thank You For Your Time!

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